1. For $x \in \mathbb{R}$, the floor of x is defined by $\lfloor x \rfloor := \max\{n \in \mathbb{Z} : n \le x\}$. Determine the point of continuity:

(a) $f(x) := \lfloor x \rfloor$.

(b) $h(x) := \lfloor \frac{1}{x} \rfloor$

2. Give an example for each of the following:

(a) $f: \mathbb{R} \to \mathbb{R}$ continuous only at one point.

(b) $f: \mathbb{R} \to \mathbb{R}$ discontinuous everywhere but $|f|$ continuous everywhere

(c) $f: \mathbb{R} \to \mathbb{R}$ continuous on $\mathbb{R} \setminus \mathbb{Q}$ but discontinuous on $\mathbb{Q}$.

3. Let $A \subset \mathbb{R}$ and let $f: A \to \mathbb{R}$ be a continuous at a point $c \in A$. Show that $\forall \varepsilon > 0$, $\exists V_\delta(c)$ s.t. $x, y \in A \cap V_\delta(c)$ then $|f(x) - f(y)| < \varepsilon$.

4. Let E be a non-empty subset of $\mathbb{R}$. For $x \in \mathbb{R}$, define $f_E(x) := \inf\{|x-y| : y \in E\}$. Show that f_E is well-defined and is Lipschitz on $\mathbb{R}$.

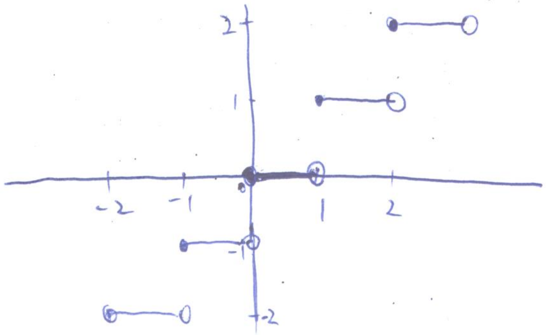
5. Suppose that $f: \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $\lim_{x \to \infty} f(x) = 0 = \lim_{x \to -\infty} f(x)$. Prove that f is bounded on $\mathbb{R}$ and attains either a maximum or minimum on $\mathbb{R}$. Give an example to show that both a maximum and a minimum need not to be attained.

6. Let $I = [a, b]$, let $f : I \to \mathbb{R}$ be continuous on I, and assume that $f(a) < 0 < f(b)$.

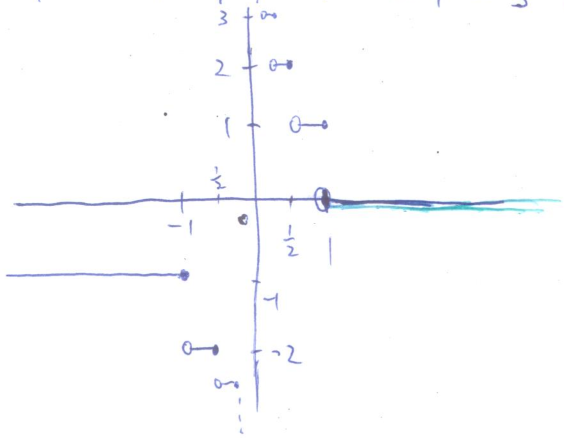
Let $W := \{x \in I : f(x) < 0\}$, and let $w := \sup W$.

Prove that $f(w) = 0$

1 a) continuous if $x \neq 0, \pm 1, \pm 2, \ldots$



b) continuous if $x \neq 0, \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \ldots$



2 a) $f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

$f(x)$ is continuous only at $x = 0$

b) $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

$f(x)$ is discontinuous everywhere.

$|f(x)| = 1 \quad \forall x \in \mathbb{R}$. $|f|$ is continuous everywhere

c) $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{q} & \text{if } x \text{ is rational}, \ x = \frac{p}{q} \text{ in lowest terms}, q > 0 \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

$f(x)$ is continuous on $\mathbb{R} \setminus \mathbb{Q}$ but distinous on $\mathbb{Q}$

3. Definition: f continuous at $c \Leftrightarrow$

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $x \in A$, $|x-c| < \delta$, then $|f(x) - f(c)| < \varepsilon$

Let $\varepsilon > 0$, $\exists V_\delta(c)$ s.t. if $x \in A \cap V_\delta(c)$, then $|f(x) - f(c)| < \frac{\varepsilon}{2}$

if $y \in A \cap V_\delta(c)$, then $|f(y) - f(c)| < \frac{\varepsilon}{2}$

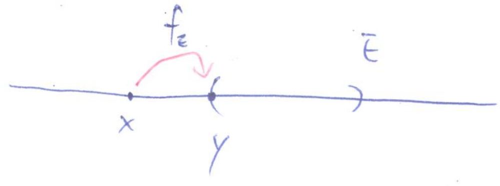
Hence $|f(x) - f(y)| = |f(x) - f(c) + f(c) - f(y)|$

$\leq |f(x) - f(c)| + |f(y) - f(c)|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$< \varepsilon$

4.



Since E is non-empty set, $\exists z \in E$.

Also, $|x - z| < \infty$.

Since $z \in E$ and $|x-z| < \infty$, $\{|x-y| : y \in E\}$ is not empty set

and $0 \leq \inf\{|x-y| : y \in E\} < |x-z| < \infty$

Then f_E is well-defined

Definition: f is Lipschitz if let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$.
If $\exists K > 0$ s.t. $|f(x) - f(u)| \leq K|x-u|$ $\forall x, u \in A$.

Want to show $\forall x, u \in \mathbb{R}$, $f(x) - f(u) \leq K|x-u|$

$f(u) - f(x) \leq K|x-u|$ for some $K > 0$

Since $f(u) = \inf\{|u-y| : y \in E\}$

let $\varepsilon > 0$, $f(u) + \varepsilon > |u - y|$ for some $y \in E$

4. Since $y \in E$, $f(x) \leq |x-y|$

Then $f(x) - f(u) - \varepsilon < |x-y| - |u-y|$

$\leq |x-u|$ triangle inequality

$f(x) - f(u) < |x-u| + \varepsilon$

Since ε is arbitrary, $f(x) - f(u) \leq |x-u|$

reverse x and u, $f(u) - f(x) < |u-x| = |x-u|$

then $|f(x) - f(u)| < |x-u|$

f is Lipschitz and hence continuous.

5. If $f = 0$ for all $x \in \mathbb{R}$. Then f is bounded and attains both max. and min. at 0.

If $f \neq 0$. $\exists\, t \in \mathbb{R}$ s.t. $|f(t)| \neq 0$

Since $\lim_{x\to\infty} f(x) = 0 = \lim_{x\to-\infty} f(x)$, Choose $\varepsilon = \frac{1}{2}|f(t)|$

$\exists\, K_1 > 0$ s.t. if $x > K_1$, $|f(x)| < \frac{1}{2}|f(t)|$

$\exists\, K_2 > 0$ s.t. if $x < -K_2$, $|f(x)| < \frac{1}{2}|f(t)|$

Choose $K = \max\{K_1, K_2\}$, then if $|x| > K$, $|f(x)| < \frac{1}{2}|f(t)|$

since $|f(t)| > \frac{1}{2}|f(t)|$, $|t| \leq K$

Let $I = [-K, K]$, by Boundness Thm, f is bounded on I

since $|f(x)| < \frac{1}{2}|f(t)|$ $\forall |x| > K$, f is bounded on $\mathbb{R} \setminus I$

Thus, f is bounded on $\mathbb{R}$

5. Also by Max-Min Thm, f attain max. and min on I

Then $\exists\ x^*, x_* \in I$ s.t. $f(x^*) = \sup f(I)$, $f(x_*) = \inf f(I)$

if $f(t) > 0$, $f(x^*) \geq f(t) > \frac{1}{2} f(t) > f(x)$ $\forall x \in \mathbb{R} \setminus I$

$f(x^*)$ is max of f on $\mathbb{R}$

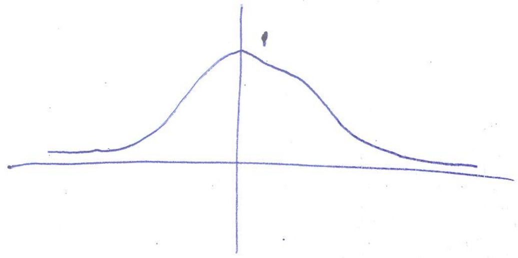
f attain max.

if $f(t) < 0$, $f(x_*) \leq f(t) < \frac{1}{2} f(t) < f(x)$ $\forall x \in \mathbb{R} \setminus I$

$f(x_*)$ is min of f on $\mathbb{R}$

f attain min.

e.g. $f(x) = \dfrac{1}{x^2+1}$,



only max. is attained.

6. Since $w = \sup W$

$w - \frac{1}{n} < x_n$ for some $x_n \in W$

Then $w - \frac{1}{n} < x_n \leq w$ $\forall n$

$\lim_n w - \frac{1}{n} = w = \lim_n w$

$\Rightarrow \lim_n x_n = w$

since f is continuous, $f(x_n)$ converges to $f(w)$

$\lim_n f(x_n) = f(w)$

since $f(x_n) < 0$ $\forall n \in \mathbb{N}$

$f(w) = \lim_n f(x_n) \leq 0$

If $f(w) < 0$, by Thm 4.29, $\exists\ V_\delta(w)$ s.t. $f(x) < 0$ $\forall x \in V_\delta(w)$

Then $w + \frac{\delta}{2} > w$

and $f(w + \frac{\delta}{2}) < 0$

Thus $w + \frac{\delta}{2} \in W$

contradiction

$\Rightarrow f(w) = 0$